Generalized synchronization in coupled Ginzburg-Landau equations and mechanisms of its arising

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Generalized chaotic synchronization regime is observed in the unidirectionally coupled one-dimensional Ginzburg-Landau equations. The mechanism resulting in the generalized synchronization regime arising in the coupled spatially extended chaotic systems demonstrating spatiotemporal chaotic oscillations has been described. The cause of the generalized synchronization occurrence is studied with the help of the modified Ginzburg-Landau equation with additional dissipation.

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Chaotic synchronization is one of the fundamental phenomena actively studied recently [1,2], having both important theoretical and applied significance (e.g., used for information transmission by means of deterministic chaotic signals [3,4], in biological [5] and physiological [6] tasks, for controlling of lasers [7,8] and microwave systems [9], etc.). Recently, several types of chaotic synchronization have been observed in coupled nonlinear oscillators. These are the phase synchronization [10], generalized synchronization [11], lag synchronization [12], intermittent lag [13], and intermittent generalized [14] synchronization behavior, complete synchronization [15].

All synchronization types mentioned above are associated with each other (see, for detail, Refs. [16–19]), but the relationship between them has not been completely clarified yet. In particular, in our works [18–20] it was shown that the phase, generalized, lag, and complete synchronization are closely connected with each other and, as a matter of fact, they are different manifestations of one type of synchronous oscillation behavior of coupled chaotic oscillators called the time-scale synchronization. For each type of synchronization there are their own ways to detect the synchronized behavior of coupled chaotic oscillators.

In the last decade synchronization of spatially extended systems demonstrating spatiotemporal chaos has attracted much interest. The possibility of the complete synchronization and phase synchronization of spatially extended systems such as coupled Ginzburg-Landau equations [2,21,22], coupled Kuramoto-Sivashinsky equations [23], arrays of coupled oscillators [24], and coupled map lattices [2] has been demonstrated recently. In particular, the experimental phase synchronization has been observed for a plasma discharge tube in Ref. [25]. In our work [9] we have shown that the time-scale synchronization takes place in unidirectionally coupled spatially extended electron-wave systems.

One of the interesting and intricate types of the synchronous behavior of unidirectionally coupled chaotic oscillators is the generalized synchronization [11]. The presence of the generalized synchronization between the response $\mathbf{x}_r(t)$ and

drive $\mathbf{x}_d(t)$ chaotic systems means that there is a functional relation $\mathbf{x}_r(t) = \mathbf{F}[\mathbf{x}_d(t)]$ between system states after the transient is finished. This functional relation $\mathbf{F}[\cdot]$ may be smooth or fractal. According to the properties of this relation, the generalized synchronization may be divided into the strong synchronization and weak synchronization, respectively [26]. There are several methods to detect the presence of the generalized synchronization between chaotic oscillators, such as the auxiliary system approach [27] or the method of calculating the conditional Lyapunov exponents [26].

In this work we have used the auxiliary system approach proposed first in Ref. [27]. We consider the dynamics of the drive $\mathbf{x}_d(t)$ and response $\mathbf{x}_r(t)$ systems. At the same time we also consider the dynamics of the auxiliary system $\mathbf{x}_a(t)$ which is identical to the response system $\mathbf{x}_r(t)$ but starts with the other initial conditions, i.e., $\mathbf{x}_r(t_0) \neq \mathbf{x}_a(t_0)$. In the absence of the generalized synchronization between the drive $\mathbf{x}_d(t)$ and response $\mathbf{x}_r(t)$ systems, the phase trajectories of the response $\mathbf{x}_r(t)$ and auxiliary $\mathbf{x}_a(t)$ systems share the same chaotic attractor but are unrelated. In the case of the generalized synchronization the behavior of the response $\mathbf{x}_r(t)$ and auxiliary $\mathbf{x}_a(t)$ systems becomes identical after the transient dies out (it may take much time [14]) due to the generalized synchronization relations $\mathbf{x}_r(t) = \mathbf{F}[\mathbf{x}_d(t)]$ and $\mathbf{x}_a(t) = \mathbf{F}[\mathbf{x}_d(t)]$. Obviously, in the case of the generalized synchronization the condition $\mathbf{x}_d(t) = \mathbf{x}_r(t)$ should be satisfied and the identity of the response and auxiliary systems is a simpler criterion to test the presence of the generalized synchronization rather than finding the unknown functional relationship $\mathbf{F}[\cdot]$.

Note, that the generalized synchronization has been studied in detail only for the chaotic systems with a few degrees of freedom and for the discrete maps [11,26,27]. In particular, in Ref. [28] we have shown that the behavior of the response chaotic system in the regime of the generalized synchronization is equal to the dynamics of the modified system (with the additional dissipation) under the external chaotic force. However, the generalized synchronization of the spatially extended chaotic systems has not been studied in detail. Here we note only Ref. [29] in which the occurrence of the generalized synchronization in the spatially extended model describing a chemical reaction has been found. The mechanism of the establishment of the generalized synchronization in the spatially extended chaotic systems is also unclear.

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In this paper we study numerically the generalized synchronization of the unidirectionally coupled complex Ginzburg-Landau equations (CGLE's). The Ginzburg-Landau equation (GLE) is a fundamental model for the pattern formation and turbulence description. This equation is used frequently to describe many different phenomena in laser physics [30], chemical turbulence [31], fluid dynamics [32], bluff body wakes [33], etc. (see also Ref. [34]).

Let us consider two one-dimensional CGLE's [21,34,35] coupled unidirectionally,

$$\frac{\partial v}{\partial t} = v - (1 - i\alpha_d)|v|^2 v + (1 + i\beta_d)\Delta v, \quad v \in [0,L], \quad (1)$$

$$\frac{\partial u}{\partial t} = u - (1 - i\alpha_r)|u|^2 u + (1 + i\beta_r)\Delta u + \varepsilon(v - u), \quad u \in [0,L]$$

$$(2)$$

with periodical boundary conditions. Equation (1) describes the drive system and Eq. (2) corresponds to the response one. In our investigation the parameters of the drive systems are chosen as $\alpha_d = 1.5$, $\beta_d = 1.5$. To study the generalized synchronization of the nonidentical systems we have chosen the different values of control parameters ($\alpha_r = 4.0$ and $\beta_r = 4.0$) for the response system (2). The choice of such values of the control parameters results in the autonomous systems being in the spatiotemporal chaotic regime. Parameter ε determines the strength of the unidirectionally dissipative coupling between the response and drive systems, the interaction of them being in each point of space. For $\varepsilon = 0$, Eqs. (1) and (2) describe two uncoupled complex fields $u(x,t)$, $v(x,t)$, each of them obeying an autonomous GLE.

All calculations were performed for a fixed system length $L = 40\pi$ and random initial conditions. The numerical code was based on a semiimplicit scheme in time with finite differences in space. In all simulations we used a time step $\Delta t = 0.0002$ for the integration and a space discretization $\Delta x = L/1024$ (1024 mesh points).

With the growth of the coupling strength ε the generalized synchronization between considered systems arises. The value of the coupling strength corresponding to the onset of the generalized synchronization is $\varepsilon = \varepsilon_{GS} \approx 0.75$. We detected the presence of the generalized synchronization between unidirectionally CGLE's with the help of the auxiliary system approach [27]. As the auxiliary system $u_a(x,t)$, we consider the media described by GLE (2) which is identical to the response system $u(x,t)$ but starts with the other initial spatial distribution, i.e., $u_a(x,t_0) \neq u(x,t_0)$. Figure 1 shows the spatiotemporal distributions of the module of the difference between the states of the response and auxiliary systems $|u(x,t) - u_a(x,t)|$ for cases of the absence [Fig. 1(a) small value of coupling strength $\varepsilon < \varepsilon_{GS}$] and the presence [Fig. 1(b), value of coupling strength $\varepsilon > \varepsilon_{GS}$] of the generalized synchronization regime. In this figure one can see, that in the second case the difference of the states of the response and auxiliary systems in every point of space tends to be zero after coupling begins, which means the presence of the generalized synchronization between the drive and response CGLE's.
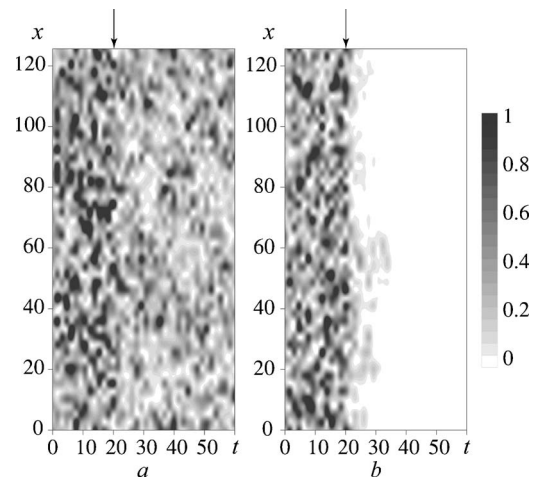

FIG. 1. (Color online) The dependence of the module of the difference between the states of the response and auxiliary systems $|u(x,t) - u_a(x,t)|$ for cases of absence (a) and presence (b) of the generalized synchronization on time t and space x. The coupling parameter between the drive and response systems has been selected as $\varepsilon = 0.4$ in (a) and $\varepsilon = 0.9$ in (b). The time moments marked by arrows correspond to the coupling switching-on between the drive and response systems.

To explain the mechanism of the generalized synchronization arising, following Ref. [28], we consider the dynamics of the response system (2) as the nonautonomous dynamics of a modified spatially extended system,

$$\frac{\partial u_m}{\partial t} = u_m - (1 - i\alpha_r)|u_m|^2 u_m + (1 + i\beta_r)\Delta u_m - \varepsilon u_m,$$

$$u_m \in [0,L], \quad (3)$$

under the external force (εv). Note, that the term $-\varepsilon u_m$ brings the additional dissipation into the modified GLE (3).

So, the control parameter ε increase may be considered as a result of two cooperative processes taking place simultaneously. The first of them is an increase of the amplitude of the external signal on the response system and the second one is the growth of the dissipation in the modified spatially extended system (3). As a result of the second process, in the modified system a decrease of the amplitude of chaotic oscillations is observed. At the coupling strength $\varepsilon = \varepsilon_0 = 1$ in the spatially extended system the stable homogenous spatiotemporal state is established rigidly in space and time. In Fig. 2, the dependence of the square of the amplitude of oscillations $\langle u_m^2(x,t) \rangle$ of the modified GLE (3) averaged over space and time on the parameter ε is shown (symbols ■). One can easily see, that the averaged amplitude of oscillations decreases linearly with the growth of the dissipation term $-\varepsilon u_m$ (i.e., with the increase of the coupling strength ϵ).

In work [28] it has been shown, that there are two mechanisms of the generalized synchronization arising. The first of them is determined by introducing the additional dissipation in the response system by means of the dissipative term ($-\varepsilon u_m$). If the generalized synchronization is observed in (1) and (2) the modified system displays the periodic oscillations and may undergo transition to the stable homogenous spa-
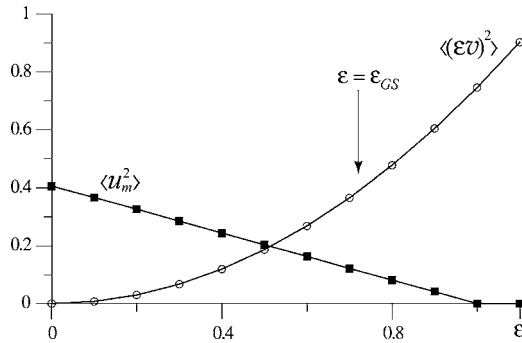
FIG. 2. The dependence of averaged power of the oscillations $\langle u_m^2(x,t) \rangle$ in the modified system ($\blacksquare$) and the external force amplitude $\langle (\varepsilon v(x,t))^2 \rangle$ ($\bigcirc$) on coupling strength ε. The value of the parameter ε_{GS} corresponding to the onset of the generalized synchronization regime is shown by an arrow.

tiotemporal state with the growth of the parameter ε (see Ref. [28]). This mechanism of the generalized synchronization arising is realized in the considered spatially extended system at $\varepsilon = \varepsilon_0 = 1$. Note, that for the considered systems the generalized synchronization regime reveals at value of the coupling strength ε_{GS} (it is marked by an arrow in Fig. 2) which is less than the value of the coupling strength ε_0 at which the stable spatiotemporal state is established (i.e., $\varepsilon_{GS} < \varepsilon_0$).

Such behavior is detemined by the second mechanism of the generalized synchronization arising [28]. Let us consider the dependence of the square of the external force amplitude $\langle (\varepsilon v^2) \rangle$ averaged over space and time, influenced on the response GLE from the drive system. This dependence is shown in Fig. 2 (symbols $\bigcirc$). Figure 2 shows that the power of the drive signal effecting the response system increases rapidly with the coupling parameter growth. As a result, for $\varepsilon = \varepsilon_{GS}$, the power of the external force exceeds the level of proper oscillations of the response system approximately by 3 times. It is clear, that in this case the great external force moves the spatiotemporal state of the response system into the regions of the phase space with the strong dissipation. So, proper spatiotemporal chaotic dynamics of the modified system (modified GLE) appears to be suppressed and the generalized synchronization is observed for $\varepsilon_{GS} < \varepsilon_0$. It is important to note, that in the range of the coupling strength $\varepsilon \in (\varepsilon_{GS}, \varepsilon_0)$ the generalized synchronization arising is caused by the simultaneous action of two mechanisms, each of them brings the contribution to the establishment of the synchronous regime.

So, in the considered spatially extended system the generalized synchronization arising is determined by two mechanisms taking place simultaneously which causes the suppression of proper chaotic spatiotemporal oscillations by means of the additional dissipation introduced in the spatially extended active system. In the case of unidirectionally coupled CGLE the arising of the generalized synchronization
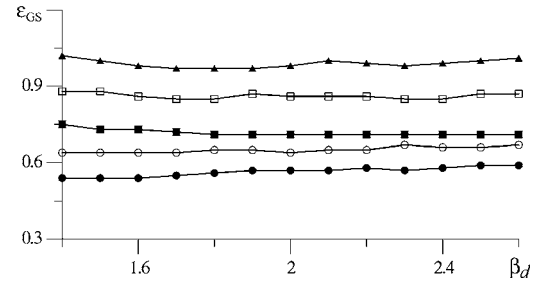

FIG. 3. The dependence of the coupling strength ε_{GS} corresponding to the onset of the generalized synchronization regime on the drive system control parameter β_d for the different values of the control parameters $\alpha_r = \beta_r = 3.0$ ($\bullet$), 3.5 ($\bigcirc$), 4.0 ($\blacksquare$), 5.0 ($\square$), 6.0 ($\blacktriangle$) of the response system.

regime is caused by the following mechanisms. First, there is the additional dissipative terms which results in a decrease of the magnitude of own oscillations in the response spatially extended active system. Second, we observed that the great external signal destroys proper dynamics of the response system and its phase state is moved into the regions of the phase space with the strong dissipation. At the same time the simultaneous decrease of the amplitude of proper oscillations takes place due to the first mechanism discussed above.

The last means, that the coupling strength ε_{GS} corresponding to the onset of the generalized synchronization regime in the spatially extended chaotic systems should not depend strongly on the parameters of the drive system and (first of all) should be defined by the properties of the modified response system. This statement is illustrated in Fig. 3, where the dependence of the coupling strength ε_{GS} corresponding to the onset of the generalized synchronization regime on the drive system control parameter β_d and for the different values of the control parameters α_r and β_r of the response system is shown. One can see that changing the drive system parameters does not effect practically on the threshold ε_{GS} of the generalized synchronization arising in the response system with growth of the coupling strength ϵ. It confirms the consideration of the mechanisms of the generalized synchronization regime arising in the coupled CGLE's.

In conclusion, we have explained the generalized synchronization arising in the unidirectionally coupled CGLE's and shown that the generalized synchronization in spatially extended chaotic systems is determined by the additional dissipation introduced into the response system. In this case the coupling parameter increase is equivalent to the simultaneous growth of the dissipation and the amplitude of the external signal.

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